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Instanton molecules at high temperature — the Georgi-Glashow model and beyond

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Abstract: We show that correlators of some local operators in gauge theories are sensitive to the presence of the instantons even at high temperature where the latter are bound into instanton-anti-instanton "molecules". We calculate correlation functions of such operators in the deconfined phase of the 2+1 dimensional Georgi-Glashow model and discuss analogous quantities in the chirally symmetric phase of QCD. We clarify the mechanism by which the instanton-anti-instanton molecules contribute to the anomaly of axial U(1) at high temperature.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons, Confinement, QCD.

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1. Introduction

In this paper we examine one particular aspect of high temperature phase of gauge theories. Specifically, many gauge theories at zero temperature have instantons. The behavior of the instanton ensemble is crucial for understanding of many dynamical properties of these theories.

In QCD with massless fermions there are indications that the instanton "liquid" is responsible for the chiral symmetry breaking [1]. A successful phenomenology of chiral symmetry breaking has indeed been developed based on this idea. Another example of a gauge theory that supports instantons is the 2+1 dimensional Georgi-Glashow (GG) model. In this theory (as well as in 2+1 D compact electrodynamics) the instanton-monopole gas has been shown analytically to be responsible for confinement a long time ago [2].

When heated above a certain critical temperature these gauge theories undergo a phase transition. In QCD the chiral symmetry is restored, while in the GG model the deconfining transition occurs. The status of the instantons in the hot phase is less certain. One thing is clear, they become less important at high temperature. The temperature acts as an infrared cutoff on the instanton size in QCD, suppressing the instantons of the size greater than the inverse electric Debye screening mass [3].

There is also another, perhaps more significant effect. In the GG model it has been shown analytically [4, 5] that the instantons at high temperatures are bound into "molecules". The same is believed to be true in QCD [6].[1]

This "disappearance" of instantons at high temperature has been discussed in the literature. Naively one may expect that once the instantons are bound in pairs, their effect in the infrared is negligible. If this is the case, the anomaly of the axial $\mathrm{U}_A(1)$ symmetry of QCD disappears, and the chiral symmetry of QCD in the hot phase is enhanced to $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}_A(1)$. One dynamical effect of such symmetry enhancement would be the degeneracy of the correlation lengths in the scalar (δ and σ) and the pseudo-scalar (π and η) channels. In fact various references to the restoration of the axial $\mathrm{U}_A(1)$ symmetry can be found in the literature [8].

Of course this argumentation is too naive, and is recognized as such by many authors in [8]. The statement about the anomaly of $\mathrm{U}_A(1)$ in QCD, $\partial_\mu J^\mu_A = (N_f/16\pi^2) \times \tilde{F}F$, is an operatorial one. As an operatorial statement it remains true at nonzero temperature. Thus the effects of the anomalous breaking are bound to be seen at some level even at high temperature. Indeed the lattice data does show that the scalar and pseudo-scalar correlation lengths are not degenerate[9]. Some analytic discussion of the consequences of the anomaly at high temperature has been given in [10]. In particular it has been shown that in the two flavor case the instantons contribute directly to the correlators of the fermionic bilinears, and thus very likely lead to splitting between the scalars and the pseudo-scalars. For $N_f > 2$ no such direct contributions exist. Thus the term "$\mathrm{U}_A(1)$ restoration" is not to be taken literally. Still since no analytic calculation of the instanton effects at high temperature are available, one may have lingering doubts about their importance. It would thus be useful to have an explicit analytic calculation of some quantities which are most sensitive to the anomaly, and thus exhibit it in a clear way at high temperature.

In QCD this is a difficult task due to the large number of degrees of freedom associated with an instanton (color orientations). However in the Georgi-Glashow model situation is much simpler, since the instantons are not degenerate. The theory is weakly interacting at high temperature, and thus under analytic control. Moreover, in many respects the Georgi-Glashow model is very similar to QCD, especially as far as the role of instantons is concerned. The purpose of this paper is therefore to present explicit calculations of the instanton effects at high temperature in the Georgi-Glashow model and the close analogy between these results and our expectations in QCD.

[1] It has been even suggested that the binding of the instantons drives the chirally restoring phase transition [6] This last point has not been definitively proven however. Indeed, this is known not to be the case in the GG model [5], and a physically different mechanism for the chiral transition was suggested in [7]. Our interest here however is not in the phase transition itself, but rather in the properties of the hot phase. In this respect both approaches of [6] and [7] agree that the instantons are bound in "molecules".

The general pattern of symmetries in the theories we have in mind is the following. On the classical level the action is invariant under a global symmetry group G. Quantum effects break this symmetry anomalously down to its subgroup H. At zero temperature H is spontaneously broken. At high temperature H is restored, but G is still broken by the anomaly. Thus a correlator O, which is invariant under H, but not invariant under G, is allowed to be non-vanishing in the anomalous theory, but has to vanish if the anomaly of G were to disappear. This particular set of correlators is therefore likely to be very sensitive directly to the instanton effect. We will consider examples of such operators in the following.

There are many parallels between QCD with massless fermions and the 2+1 GG model. In particular the confining physics of the latter is in many respects similar to the chiral physics of QCD. These parallels been discussed in detail in [7]. The basic correspondences are the following.

• Classical axial $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}_A(1)$ symmetry of QCD $\leftrightarrow$ Classical magnetic $\mathrm{U}_M(1)$ symmetry of the GG model .

• Axial anomaly due to instantons in QCD$\leftrightarrow$ Magnetic anomaly due to monopoles in the GG model.

• Non-anomalous $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ subgroup of $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}_A(1)$ in QCD $\leftrightarrow$ Non-anomalous Z_2 subgroup of $\mathrm{U}_M(1)$ in the GG model.

• Spontaneous breaking of $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ by the chiral condensate $\langle\bar{\psi}\psi\rangle$ in QCD$\leftrightarrow$ Spontaneous breaking of Z_2 by the vortex condensate $\langle V\rangle$ in the GG model.

Keeping this analogy in mind, we proceed to analyze in the next section the monopole-instanton effects in the 2+1D GG model at high temperature. Since the theory is weakly interacting, the calculations here are under complete analytic control. In section 3 we calculate explicitly multi-instanton contributions to relevant correlation functions. Our results illustrate very clearly the effects that we also expect to take place in QCD. We conclude in section 4 with discussing further the analogy with QCD.

2. The instantons and their interactions in the Georgi-Glashow model at high temperature

2.1 The model

Consider the SU(2) gauge theory with a scalar field in the adjoint representation in 2+1 dimensions.

$$S = -\frac{1}{2g^2}\int d^3x \mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \int d^3x \left[\frac{1}{2}(D_\mu h^a)^2 + \frac{\lambda}{4}(h^a h^a - v^2)^2\right]. \tag{2.1}$$

In the weakly coupled regime $v^2 \gg g^2$, perturbatively the gauge group is broken to U(1) by the large expectation value of the Higgs field. The photon associated with the unbroken subgroup is massless whereas the Higgs and the other two gauge bosons $W^{\pm}$ are heavy with the masses

$$M_H^2 = 2\lambda v^2 \,, M_W^2 = g^2 v^2 \,. \tag{2.2}$$

Thus perturbatively the theory behaves very much like electrodynamics with spin one charged matter.

This theory has a global symmetry - the magnetic symmetry [11, 12]. Classically the following gauge invariant current is conserved

$$\tilde{F}^\mu = \tilde{F}^\mu_a n^a - \frac{1}{g}\epsilon^{\mu\nu\lambda}\epsilon^{abc} n_a (\mathcal{D}_\nu n)^b (\mathcal{D}_\lambda n)^c \tag{2.3}$$

with n^a a unit vector in the color space in the direction of the Higgs field, $n^a = h^a/|h|$. This defines a conserved charge through $\Phi = \int d^2 x B$. This continuous symmetry is spontaneously broken in the vacuum, and the massless photon is the Goldstone boson which reflects this breaking in the spectrum.

However as is well known the classical action eq. (2.1) supports instanton-monopole solutions with finite euclidean action

$$\begin{aligned} h^a(\vec{x}) &= \hat{x}^a h(r) \\ A^a_\mu(\vec{x}) &= \frac{1}{r}\left[\epsilon_{a\mu\nu}\hat{x}^\nu(1-\phi_1) + \delta^{a\mu}\phi_2 + (rA - \phi_2)\hat{x}^a\hat{x}_\mu\right] \end{aligned} \tag{2.4}$$

where $\hat{x}^a = x^a/r$. In the presence of such a monopole the magnetic current is not conserved, but rather has a non-vanishing divergence proportional to the monopole density:

$$\partial_\mu \tilde{F}_\mu = \frac{4\pi}{g}\rho \tag{2.5}$$

The $\mathrm{U}_M(1)$ magnetic symmetry is thus *anomalous* in the quantum theory. As discussed in detail in [12] only the discrete Z_2 subgroup is unaffected by anomaly and thus remains a symmetry in the full quantum theory.

The local order parameter that transforms as an eigen-operator under the Z_2 magnetic symmetry is the magnetic vortex operator V:

$$V \to -V \tag{2.6}$$

The operator V has been constructed explicitly in [11, 12]

$$\begin{aligned} V(x) &= \exp\frac{i}{g}\int d^2y \,\left[\epsilon_{ij}\frac{(x-y)_j}{(x-y)^2} n^a(y) E^a_i(y) + \Theta(x-y) J_0(y)\right] \\ &= \exp\frac{2\pi i}{g}\int_C dy^i \epsilon_{ij} n^a E^a_j(y) \end{aligned} \tag{2.7}$$

One can think of it as a singular SU(2) gauge transformation with the field dependent gauge function

$$\lambda^a(y) = \frac{1}{g}\Theta(x-y) n^a(\vec{y}) \,. \tag{2.8}$$

The dynamical effects of the anomaly are twofold. First, the photon becomes a pseudo-Goldstone boson and acquires a finite albeit small mass $m_{ph}^2 \propto \exp\{-4\pi M_W/g^2\}$ (in the BPS limit) . Another effect is confinement of $W^\pm$ bosons with the string tension $\sigma \propto g^2 m_{ph}$.

When heated the model undergoes a phase transition to deconfined phase at $T_c = g^2/4\pi$ [5]. At this transition the non-anomalous magnetic Z_2 symmetry is restored.

2.2 The effective action and the instanton interactions

We are interested primarily in the high temperature phase, $T > T_c$. At these temperatures the dimensional reduction applies and the effective lagrangian was derived in [5] in terms of the abelian Polyakov line P.

$$S = \int d^2x \left\{ \frac{2T}{g^2}(\partial_i P)(\partial_i P^*) - \frac{T}{2g^2} M_D^2 (P^2 + P^{*2}) + \frac{a^2}{4\pi^2} \ln\left(\zeta a^2\right) \left(\epsilon_{ij}\partial_i P \partial_j P^*\right)^2 \right\} \tag{2.9}$$

Here the Polyakov line P is a pure phase $P = \exp\{i(g/T)A_0\}$, and A_0 is the component of the vector potential parallel to the direction of the Higgs field n[2].

The action eq. (2.9) deserves some explanation. Firstly, as any dimensionally reduced lagrangian it is valid at distance scales greater than the inverse temperature. The parameter a appearing in it is the ultraviolet cutoff, and is of this order $a = 2\pi/T$. The second term expanded to order A_0^2 gives the electric Debye screening mass to A_0. The mass squared is proportional to the fugacity of the charged particles, which in this model are the $W^\pm$ bosons $M_D^2 \propto \exp\{-M_W/T\}$. The exact proportionality coefficient is straightforwardly calculable at one loop, but is inessential for our purposes. The last term is induced by the monopole-instantons. As discussed in [5] the monopoles in this effective action appear as vortices of the Polyakov line P with unit winding. Indeed the last term in eq. (2.9) is of the type of Skyrme term and vanishes on any smooth configuration of A_0. Its only contribution comes from points at which the winding of P is concentrated. It therefore just counts the number of vortices and anti-vortices of P, that is of monopoles and anti-monopoles. On a configuration with one monopole the contribution of this term to the partition function is equal to ζ. Thus ζ is the fugacity of the monopole (or anti-monopole), $\zeta \propto \exp\{-4\pi M_w/g^2\}$ [3]. An important point is that this dimensionally reduced ac-

[2]Note that the transition temperature is much less than the Higgs expectation value $T_c \ll v^2$. Thus the direction of the Higgs field is well defined also above the transition temperature. Of course at high enough temperature $T = O(v^2)$ the Higgs field will also fluctuate strongly and the nonabelian part of the gauge group will also become important. At these temperatures the theory becomes essentially nonabelian with all the ensuing complications. To keep our calculation under analytic control we restrict ourselves to temperatures $T_c < T < v^2$.

[3]The proportionality constant is slightly different than at zero temperature reflecting the fact that the field modes with momenta $p > T$ have been integrated out to arrive at the effective action eq. (2.9) [4].

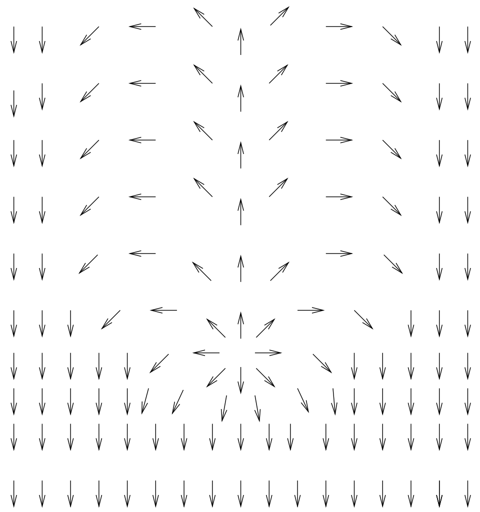

Figure 1: The string-like configuration of the Polyakov line P that accompanies the monopole-instanton in the high temperature phase.

tion has not been derived in the derivative expansion, but rather in the expansion in powers of the exponentially small parameters $M_D a$ and ζ [5]. It is thus valid for all distance scales above the UV cutoff a.

It is easy to see from the effective action eq. (2.9) that the monopoles are bound by a (screened) linear potential. Consider a configuration with unit winding of the field P. Due to the potential term P^2 the minimal action configuration can not be a rotationally symmetric hedgehog. Such a configuration would "cost" action proportional to the volume, since the field P would be away from its vacuum value everywhere in space. The best one can do is to have a quasi one dimensional strip in which the winding is concentrated, while everywhere else in space P would be equal to 1 (or -1). This configuration is schematically depicted on figure 1. The width of the "dual confining string" must be clearly of order of M_D^{-1}, while the action per unit length $s \propto (T/g^2)M_D$. Thus the action of a single monopole diverges linearly with the size of the system. Obviously the monopole-anti-monopole pair separated by a distance L have the action Ls. When the distance is large enough, another pair can be popped from the thermal ensemble and screen the linear potential. The critical distance at which this happens is clearly determined by[4]

$$L_c s = 2\log\zeta = \frac{8\pi M_W}{g^2} \tag{2.10}$$

or

$$L_c \propto \frac{M_W}{T} M_D^{-1}\,. \tag{2.11}$$

[4]In this equation we neglected the effects of the finite temperature "renormalization" of the monopole fugacity ζ.

Thus as long as the temperature is much lower than M_W, the length of the "dual string" is much greater than its thickness. The "potential" between the monopoles is therefore linear, but screened at large distances, much in the same way as the confining potential in gauge theories with heavy fundamental charges.[5]

Now it is interesting to consider the correlation functions of the vortex operators. There is a direct analogy between these correlation functions and the correlators of the fermionic bilinears in QCD. In particular if the magnetic $\mathrm{U}_M(1)$ were restored at high temperature, this would imply a simple relation between correlation lengths in different channels, directly analogous to the would be degeneracy between the η and the pions in QCD. Recall that the vortex operator in the GG model is a pseudo-scalar [12], so that the parity transformation acts on it as

$$V \to V^* . \tag{2.12}$$

The parity even and odd eigenstates are therefore

$$V^+ = \frac{V + V^*}{2}, \qquad V^- = i\frac{V - V^*}{2} . \tag{2.13}$$

If the magnetic $\mathrm{U}_M(1)$ is restored, the correlators $\langle VV\rangle$ and $\langle V^*V^*\rangle$ both vanish, and thus the following relation should hold

$$\langle V^+(x)V^+(y)\rangle = \langle V^-(x)V^-(y)\rangle . \tag{2.14}$$

The correlation lengths in the parity even and the parity odd channels should be the same. This is the direct analog of the would be equality of the correlation lengths in the $\eta = \bar\psi\gamma_5\psi$ and the $\sigma = \bar\psi\psi$ channels in QCD. We will see in the following that this equality in fact does not hold due to the instanton effects.

3. Instanton effects at high temperature

3.1 $\mathrm{U}_M(1)$ invariant correlators of V

Let us remind the reader how calculation of the correlators of V proceeds in the framework of the effective action eq. (2.9). For pure gauge theories this has been discussed in detail in [15]. In the present, essentially abelian case the procedure is very similar. The vortex operator $V(x)$ in eq. (2.7) induces a singular abelian gauge transformation which forces the vector potential A_0 to jump by π across the curve C in the definition eq. (2.7). Thus the Polyakov line P is forced to change sign across C. Formally, the insertion of the operator $V(x)$ into the euclidean finite temperature path integral leads to the shift

$$\partial_i A_0 \to \partial_i A_0 - a_i \tag{3.1}$$

[5]Of course we should always keep in mind that while talking about instanton "potential" we really mean "action" and not energy. Thus physically "confinement" of instantons is very different from confinement of charged particles.

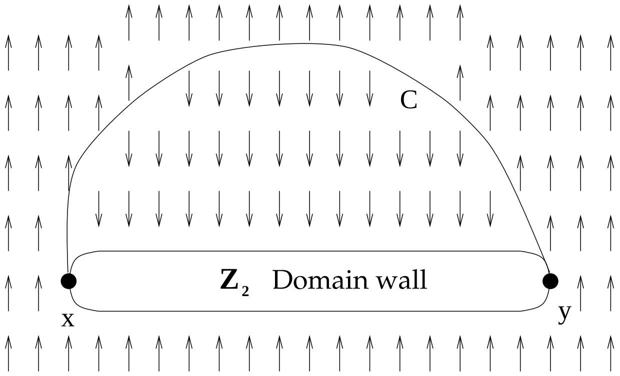

Figure 2: The configuration of the Polyakov line P that dominates the correlator $V^*(x)V(y)$.

in the derivative terms in eq. (2.9). Here a_i is a "vector potential" of a point-like vortex of vorticity $1/2$ at the point x

$$a_i(y) = \pi n_j^C(y)\delta(y - C)\,, \tag{3.2}$$

where $n^C(y)$ is a unit vector normal to the curve C at the point y. The result does not depend on the curve C, but only on its end point x, since the shape of the curve may be changed into C' by transforming $P \to -P$ in the area enclosed by $C - C'$ [15].

The calculation of the correlation function $V(x)V^*(y)$ leads to the path integral with the vortex at x and the anti-vortex at y. At high enough temperature due to the factor T/g^2 in the action this path integral can be calculated in the steepest descent approximation. The integral then is dominated by the solution of the classical equations of motion subject to the condition that P has to change sign across the curve C that connects the points x and y, see figure 2.

The shape of this solution is easy to understand [15]. In the hot phase the configurations $P = 1$ and $P = -1$ are degenerate. There are therefore classical solutions which have the form of domain walls, that interpolate between these two configurations, the so called Z_2 domain walls [16]. Since our effective action is a simple sine-Gordon theory, the solution for such a domain wall is known explicitly[6]

$$A_0 = \frac{2\pi T}{g}\arctan e^{M_D(x-x_0)}\,. \tag{3.3}$$

The action of this solution per unit length is

$$\tilde{\sigma} = \frac{8T}{g^2}M_D\,. \tag{3.4}$$

In the calculation of $V(x)V^*(y)$ the discontinuity in the boundary conditions forces P to jump from the vicinity of one "vacuum" to the vicinity of the other

[6]The Skyrme term in eq. (2.9) does not play any role in this discussion, since the solution in question is smooth and this term vanishes for smooth functions A_0.

across C. But since P has to go to the same value in all directions at infinity, it has to return to the original vacuum. The solution thus is a domain wall of figure 2. Using the result eq. (3.4) we find that for large separations[7]

$$\langle V(x)V^*(y)\rangle \propto \exp\{-|x-y|\tilde{\sigma}\}\,. \tag{3.5}$$

Actually one can do a little better and determine the prefactor quite easily. The prefactor originates from the action associated with the endpoints x and y at which the wall terminates. Close to the endpoints (at distance smaller than M_D^{-1}) the mass term is irrelevant and the configuration of A_0 must be that of a rotationally symmetric vortex (anti-vortex) of vorticity π

$$A_0(z) = \frac{\theta(z)}{2} \tag{3.6}$$

with θ a planar angle relative to the location of the vortex. Such a vortex carries the Coulomb "self energy"

$$\delta S = \frac{T}{2g^2}\pi \int_a^{M_D^{-1}} \frac{dr}{r} = -\frac{\pi T}{2g^2}\ln(aM_D)\,. \tag{3.7}$$

Adding the contribution of both end points we find[8]

$$\langle V(x)V^*(y)\rangle = (aM_D)^{\pi T/g^2}\exp\{-|x-y|\tilde{\sigma}\}\,. \tag{3.8}$$

Note that this calculation was performed entirely in the zero instanton sector. There is of course an instanton-anti-instanton contribution. We will discuss it at the end of this section.

3.2 Instanton sensitive correlators

Let us now consider the calculation of $\langle V^2(0)\rangle$, which is the simplest operator not invariant under the magnetic $\mathrm{U}_M(1)$, but still invariant under the non-anomalous Z_2. Neglecting the instanton contributions, we should find that the VEV of this operator vanishes. Indeed the path integral for $\langle V^2(0)\rangle$ has the boundary condition imposing vorticity 2π of P at the point $x=0$. In the zero instanton sector the action with such boundary condition diverges. The reason is precisely the same as the one for linear potential between the instantons. The external vortex associated with $V^2(0)$ will pick up a "dual confining string", which carries finite action per unit length. This configuration has action proportional to the linear size of the system and thus the VEV of V^2 vanishes exponentially in the thermodynamic limit. The

[7] Again the Skyrme term does not contribute, since the contribution of the cut C is subtracted explicitly by the shift eq. (3.1).

[8] In principle we should also subtract $2M_D^{-1}$ from the length of the string for consistency. This correction is however sub-leading, since it brings in a factor $\exp\{2\tilde{\sigma}M_D^{-1}\}$ which is of order 1 in the interesting temperature range.

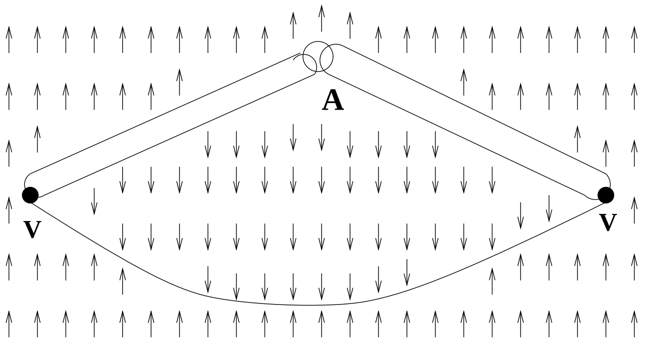

Figure 3: The configuration of one anti-instanton contributing to the correlator $V(x)V(y)$.

leading non-vanishing contribution to $\langle V^2(0)\rangle$ comes from the sector with one anti-instanton . Clearly if there is an anti-instanton in the vicinity of the point $x = 0$, the dual string that originates at $x = 0$ will terminate on it and thus the result will be finite. To calculate the expectation value we note that it is dominated by the anti-instanton sitting relatively close to the point $x = 0$, since otherwise the action of the configuration is very large. On the other hand if the coordinate of the anti-instanton x_a is smaller than M_D^{-1}, the action of the classical solution is just the action of a dipole with the vector potential

$$a_0(x) = \theta(x) - \theta(x - x_a)\,. \tag{3.9}$$

The expectation value is therefore given by the integral over the quasi zero mode x_a

$$\begin{aligned}\langle V^2\rangle &= \zeta \int_{a<|x_a|<M_D^{-1}} \frac{d^2x_a}{\pi a^2} \exp\{-S(a_0)\} \\ &= \zeta a^{-2} \int d(l^2) \exp\left\{-\frac{4\pi T}{g^2} \ln\left(\frac{l^2}{a^2}\right)\right\} \\ &= \zeta \frac{g^2}{4\pi T - g^2}\left[1 - (M_D a)^{24\pi T - g^2/g^2}\right].\end{aligned} \tag{3.10}$$

Let us now consider the correlation function $\langle V(x)V(y)\rangle$. As far as the magnetic quantum numbers are concerned, this correlator is the same as the composite operator V^2, but it is more interesting since it must also exhibit some physical correlation length. The calculation of this quantity is not much different. In the zero instanton sector it vanishes for the same reason as V^2. It carries a net "vorticity" and this vorticity has to be screened by an anti-instanton in order to give a finite result. In principle the anti-instanton can be located at an arbitrary point in space. The field configuration corresponding to two external vortices associated with insertions of V and the anti-instanton is schematically depicted on figure 3.

Clearly the contribution of anti-instantons sitting at a significant distance from the straight line connecting x and y is exponentially suppressed, since the "dual confining string" in this configuration is longer. Thus the leading contribution is given by the anti-instantons sitting within the transverse distance M_D^{-1} of the straight line. The correlation function thus is

$$\langle V(x)V(y)\rangle = \zeta \int \frac{d^2z}{\pi a^2} \exp\{-S(x,y,z)\} = \zeta \int_x^y \frac{dz_1}{\pi M_D a^2} \exp\{-S(x,y,z_1)\}\,, \quad (3.11)$$

where we have taken $x = (x_1, 0)$ and $y = (y_1, 0)$. Here z is the position of the anti-instanton. As long as $|x-y| \gg M_D^{-1}$, the main contribution comes from the anti-instantons far from the endpoints x and y. For these configurations the integral over z_1 just gives $|x-y|$, and the action is the sum of the action of the string $\tilde{\sigma}|x-y|$ plus the self energies of the anti-instanton and the vortices at the endpoints x and y. Those are calculated as in eqs. (3.7, 3.8). So that finally we get

$$\langle V(x)V(y)\rangle = \zeta (aM_D)^{(3\pi T/g^2)-1} \frac{|x-y|}{a} \exp\{-|x-y|\tilde{\sigma}\}\,. \quad (3.12)$$

Interestingly enough the correlation length that appears in this correlation function is the same as the one appearing in the correlator of vortex and anti-vortex $V(x)V^*(y)$.

Thus as expected, both the VEV of V^2 and the correlator $V(x)V(y)$ are sensitive to the instantons. Both these quantities are indeed directly proportional to the fugacity of the instanton ζ. Even though in the $\mathrm{U}_M(1)$ invariant sector the instantons appear only in bound pairs, in the calculation of the $\mathrm{U}_M(1)$ non-invariant correlation functions they do appear separately. Such a single instanton (anti-instanton) binds to the external insertion that carries the explicit $\mathrm{U}_M(1)$ charge and screens this charge. This leads to non-vanishing correlation functions in the channels whose quantum numbers match those of an integer number of instantons (anti-instantons).

3.3 Multi-instanton contributions

As mentioned earlier, there are also multi-instanton contributions to the vortex correlation functions. Those become important when the separation $|x-y|$ is large enough. The reason is that although each extra instanton-anti-instanton pair is suppressed by ζ^2, at large $|x-y|$ the "entropy" is large enough and thus these contributions are not negligible.

Consider first the correlator $V(x)V^*(y)$. In order that the multi-instanton contribution not be suppressed exponentially, all instantons must be located on the straight line connecting the points x and y. Moreover, the instantons and anti-instantons must alternate, since otherwise some sections of the dual confining string connect regions of vorticity 3, and thus has much higher dual string tension. Other than these constraints, the coordinates of instantons and anti-instantons are arbitrary and have to be integrated over. The contribution of n instantons and n anti-instantons is therefore given by

$$\langle V(x)V^*(y)\rangle_n = (aM_D)^{\pi nT/g^2} \exp\{-|x-y|\tilde{\sigma}\}\zeta^{2n}(aM_D)^{4\pi nT/g^2}\pi M_D a^{2^{-2n}} \times$$
$$\times \int_x^y dy_n \int_x^{y_n} dx_n \int_x^{x_n} dy_{n-1} \ldots \int_x^{y_1} dx_1\,. \quad (3.13)$$

Here x_i and y_i are longitudinal coordinates of instantons and anti-instantons correspondingly.[9] Performing the integration and summing over n we find

$$\langle V(x)V^*(y)\rangle = (aM_D)^{\pi T/g^2} \exp\left\{-|x-y|\tilde{\sigma}\} \cosh\{\zeta(aM_D)^{(2\pi T/g^2)-1}\frac{|x-y|}{\pi a}\right\}. \quad (3.14)$$

The multi-instantons have therefore a very nontrivial effect on the vortex correlator. They "split" the correlation length in two, generating two distinct correlation lengths

$$\tilde{\sigma}_{\pm} = \tilde{\sigma} \pm \zeta(aM_D)^{(2\pi T/g^2)-1}\frac{1}{\pi a}. \quad (3.15)$$

The calculation of the multi-instanton contributions to $V(x)V(y)$ is very similar, and gives

$$\langle V(x)V(y)\rangle = (aM_D)^{\pi T/g^2} \exp\{-|x-y|\tilde{\sigma}\} \sinh\left\{\zeta(aM_D)^{(2\pi n T/g^2)-1}\frac{|x-y|}{\pi a}\right\}. \quad (3.16)$$

Again we see that the same set of correlation lengths appears in the two correlation functions.

Eqs. (3.14), (3.16) are very simple from the point of view of the parity eigenstates V^+ and V^- eq. (2.13). They can be rewritten as

$$\begin{aligned}\langle V^+(x)V^+(y)\rangle &= (aM_D)^{\pi T/g^2} \exp\{-|x-y|\tilde{\sigma}_-\} \\ \langle V^-(x)V^-(y)\rangle &= (aM_D)^{\pi T/g^2} \exp\{-|x-y|\tilde{\sigma}_+\}. \end{aligned} \quad (3.17)$$

Thus we see explicitly that the correlation lengths in the scalar and pseudo-scalar channels are not equal due to the instanton effects. This difference decreases with temperature, but in the temperature range under consideration ($T \ll M_W$) always remains finite. Recalling that $M_D^2 \propto \exp\{-M_W/T\}$ and $\zeta \propto \exp\{-4\pi M_W/g^2\}$ we find that the ratio of the difference of the inverse correlation lengths to their sum varies between

$$\frac{\tilde{\sigma}_+ - \tilde{\sigma}_-}{\tilde{\sigma}_+ + \tilde{\sigma}_-} = \exp\left\{-\frac{\pi M_W}{g^2}\right\}, \qquad T \to T_c = \frac{g^2}{4\pi} \quad (3.18)$$

and

$$\frac{\tilde{\sigma}_+ - \tilde{\sigma}_-}{\tilde{\sigma}_+ + \tilde{\sigma}_-} = \exp\left\{-\frac{5\pi M_W}{g^2}\right\}, \qquad T \to M_W. \quad (3.19)$$

Finally it is worth mentioning that the quantities that get the direct instantons contributions naturally are proportional to the instanton fugacity, but their smallness is not uniform. For example as already mentioned, the correlation lengths in the vortex-vortex channel is the same as in the vortex-anti-vortex channel. Moreover comparing the vortex-vortex correlator eq. (3.16) with the vortex-anti-vortex correlator eq. (3.14) we see that although the former is smaller at short distances , at distances of order $\pi a\zeta^{-1}(aM_D)^{1-2\pi T/g^2}$ the two are numerically practically equal.

[9] Note that the usual symmetry factor $(1/n!)(1/n!)$ does not appear in this expression due to ordering of the coordinates of the instantons and anti-instantons.

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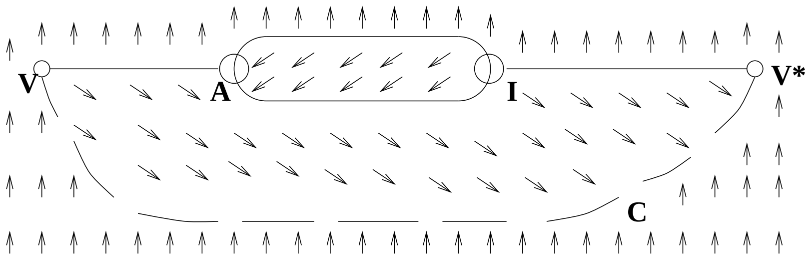

Figure 4: The instanton-anti-instanton configuration contributing to the correlator $V(x)V^*(y)$ for $N = 3$. The instanton (anti-instanton) is a source (sink) of three elementary Z_3 domain walls. The curve C is the "fake" infinitely thin wall as in eq. (3.2).

3.4 $N > 2$

Let us briefly comment on the situation in the GG type models for $N > 2$. The quantitative discussion of the phase transition will be given elsewhere [17]. Here we want to discuss only the main qualitative features. The basic degree of freedom is still a vortex operator $V(x)$, and the effective theory at high temperature is similar to eq. (2.9) with the main difference that the potential term is $P^N + P^{*N}$, and thus is Z_N invariant.[10] The instanton again is just a vortex of P with unit vorticity.

First consider the calculation of $\langle VV^*\rangle$. Just like for $N = 2$, the contribution in zero instanton sector is given by the Z_N domain wall

$$\langle V(x)V^*(y)\rangle \propto \exp\{-|x-y|\tilde{\sigma}\}\,. \tag{3.20}$$

The instanton-anti-instanton contribution now is however different. The point is that the instanton (anti-instanton), having winding number one, is the source (sink) of N Z_N domain walls. Thus the system of the vortex $V(x)$ plus an anti-instanton at some point x_1 together are the sink of $N-1$ domain walls. The string tension of this bunch is roughly $(N-1)\tilde{\sigma}$. Thus in order for the instanton-anti-instanton contribution not to be suppressed by the exponential factor, the anti-instanton must sit within a distance $[(N-2)\tilde{\sigma}]^{-1}$ of the instanton, see figure 4.

Again summing over the number of the AI pairs, we find that the contribution exponentiates

$$\begin{aligned}\langle V(x)V^*(y)\rangle = {} & (aM_D)^{(4\pi T/g^2N^2)}\exp\{-|x-y|\tilde{\sigma}\}\times \\ & \times\exp\left\{\zeta^2(aM_D)^{(2\pi T/g^2)-1}\frac{|x-y|}{\pi a^2(N-2)\tilde{\sigma}}\right\}.\end{aligned} \tag{3.21}$$

[10]The actual situation generically is more complicated than that. In particular there are $N-1$ distinct vortex operators. Also there are $N-1$ distinct abelian Polyakov lines, corresponding to the $N-1$ Cartan algebra generators, and all of those should appear in the effective action. However one can always choose the parameters of the model to be such that one of these operators is much lighter than the others. The rest of the Polyakov lines then can be integrated out and the effective action depends only on one P. For simplicity we consider only this situation in the present article, although the main conclusions are the same also in the generic case.

So the result is quite distinct from the $N = 2$ case. Only one correlation length is present, although there is indeed a direct contribution of instantons into this length of order $O(\zeta^2)$.

The behavior of the correlator $\langle V(x)V(y)\rangle$ is also very different. Since this correlation function is not symmetric under the magnetic Z_N symmetry, which is unbroken in the high temperature phase, it strictly vanishes. Instead, the instantons contribute to the correlation functions with the quantum numbers of V^N. In particular consider $\langle V(x)V^{N-1}(y)\rangle$. The leading contribution to this correlation function comes from the configuration with an anti-instanton within the distance $[(N-2)\tilde{\sigma}]^{-1}$ of the point y and an arbitrary number of the AI pairs.

$$\begin{aligned}\langle V(x)V^{N-1}(y)\rangle = \zeta(aM_D)^{[(4+N^2)\pi T/N^2g^2]-1}\exp\{-|x-y|\tilde{\sigma}\}\times \\ \times\exp\left\{\zeta^2(aM_D)^{(2\pi T/g^2)-1}\frac{|x-y|}{\pi a^2(N-2)\tilde{\sigma}}\right\}.\end{aligned} \tag{3.22}$$

In this calculation we only kept the leading contribution to the correlation function, which in the absence of instantons decays with the correlation length $l^{-1} = \tilde{\sigma}$. There is of course another contribution. The configurations where the instantons and anti-instantons are allowed to separate will lead to the component with the correlation length $l^{-1} = (N-1)\tilde{\sigma}$. This is precisely the correlation length that in the absence of instantons dominates the correlator $\langle V^{N-1}(x)V^{*(N-1)}(y)\rangle$. The instantons clearly lead to the mixing of the operators V and $V^{*(N-1)}$ in full analogy with the case $N = 2$. This mixing is accompanied with the instanton induced increase in the longest correlation length

$$[\tilde{\sigma}]^{-1} \to \left[\tilde{\sigma} - (aM_D)^{(2\pi T/g^2)-1}\frac{\zeta^2}{\pi a^2(N-2)\tilde{\sigma}}\right]^{-1}. \tag{3.23}$$

Thus for $N > 2$ the instantons do not remove the degeneracy between $V + V^*$ and $V - V^*$,but they do significantly affect other correlation functions consistently with the magnetic Z_N symmetry.

We conclude that even though at high temperature the instantons are bound in pairs, they are still relevant in the infrared. The anomaly does not disappear. Its manifestation is non-vanishing of non-singlet correlators as well as the difference of the correlation lengths in the scalar and pseudo-scalar channels for $N = 2$ and the operator mixing consistent with the unbroken magnetic Z_N for $N > 2$.

4. Discussion

The calculation presented in this paper illustrates two simple points. First, which correlation functions vanish and which don't is determined entirely by the non-anomalous symmetries. Second, the instanton contributions are important at high temperature even though the instantons in the partition function are bound in pairs. Both these features we expect of course to hold also in QCD.

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In particular let us consider the fermionic bilinear correlation functions. The fermionic bilinears that transform as irreducible representations of the non-anomalous $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ are

$$O^{ij}(x)=\bar\psi^i_L(x)\psi^j_R(x)\,,\qquad O^{\dagger ij}(x)=\bar\psi^i_R(x)\psi^j_L(x) \tag{4.1}$$

whereas the parity eigen-operators are

$$O_\pm=O\pm O^\dagger\,. \tag{4.2}$$

Now consider the correlation functions $O_+(x)O_+(y)$ and $O_-(x)O_-(y)$. If the correlators $T=\langle O(x)O(y)\rangle$ and $T^\dagger=\langle O^\dagger(x)O^\dagger(y)\rangle$ both vanish, then the correlation functions of the parity even and the parity odd operators are equal. Whether this is true or not depends on the number of flavors. For $N>2$ the vanishing of these correlation functions is assured by the $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ symmetry, since T is the product of two fundamental representations (separately for the right and left symmetries) and therefore does not contain a singlet. For $N_f=2$ on the other hand the product of two fundamentals does contain a singlet. Thus there is no symmetry obstruction for T to be nonzero.[11]

Dynamically the contributions to T come from the instanton sector. For $N_f=2$ the number of fermionic operators in T is just right to saturate the fermionic zero modes of an instanton. Recall that the fermionic zero modes at high temperature away from the core of the instanton exponentially decrease as $\exp\{-\pi T|x|\}$ [3, 1]. This is the same exponential decay as of the lowest fermionic Matsubara mode. Thus we expect the situation to be very similar to the $N=2$ Georgi-Glashow model. The perturbative behavior of the correlator $O(x)O^\dagger(y)$ is

$$\langle O(x)O^\dagger(y)\rangle\propto\exp\{-2\pi T|x-y|\}\,. \tag{4.3}$$

The instanton-anti-instanton contributions have the same exponential behavior, and extra powers of $|x-y|$ due to the arbitrariness of the instanton's (anti-instanton's) position. It is very likely that the sum over the number of instantons exponentiates as in eq. (3.14) and results in the splitting of the basic correlation length into two $2\pi T\pm x\zeta$. These two distinct correlation length then appear separately in the parity even and parity odd channels.

For $N_f>2$ the number of fermions in T does not match the number of the fermionic zero modes in the instantons, and thus the only way instantons can contribute is through AI pairs. They will however give direct contributions to the correlators of the type $T_N=\langle O(x)O_{2N-2}(y)\rangle$ where O_{2N-2} is the operator containing $2N-2$ fermions, such that the quantum numbers of T_N are the same as of 't Hooft's determinant. This is essentially the argumentation of [10]. Again this is mimicked perfectly by our discussion of the $N>2$ GG model.

[11] In this discussion we did not mention at all the anomalous $\mathrm{U}_A(1)$. Of course, if we assume that it is restored, the parity odd and even channels must be degenerate, since they transform into each other under the axial U(1).

Although analytic multi-instanton calculations in QCD are difficult, the toy GG model gives us a very clear picture of what the expected result is. Hopefully this knowledge may give us a hint of how to separate the most important contributions so that an analytic calculation may become possible after all.

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